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Dynamic Responses of Magneto-Thermo-Electro-Elastic Shell Structures with Closed-Circuit Surface Condition

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Abstract: An analytical solution for piezolaminated shell structure and embedded smart materials is presented in this study. In this study, the fundamental theory was derived based on the generic first-order transversely shearable deformation theory involving Codazzi-Gauss geometrical discretion. The fundamental equation and its boundary conditions was strenuously derived using Hamilton's principle with cooperating of Gibbs free energy function. The theory was casted in version of shell of revolution, in order to be simplified to account for commonly occurring sensors and actuator geometries and intended for wide range of common smart materials. Then the developed theory was solved by the generic forced-solution procedure. The responses and their frequency parameters were evaluated in the simply supported boundary condition. The results have shown a close agreement with those reported in literature. The developed theory and the presented solution procedure may serve as a reference in developing the magneto-thermo-electro-elastic shell theories and to improve the benchmark solutions for judging the existence of imprecise theories and other numerical approaches.

Key words: First order shear deformation theory, smart composite, smart material, structronics, piezolaminated shell, shell structures

INTRODUCTION

Structronics is concept of "Structures+Electronics", which are synergistic integration of smart, adaptive or responsive materials, that contains the main structure and the distributed functional materials (e.g., piezoelectric, piezomagnetic, electrostrictive, magnetostrictive and alike materials). Furthermore, structronic refer to a class of structures had the capability of simultaneously sensing/actuating, mechanical, electrical, magnetic and even thermal effects, as well as simultaneously generating control forces to eliminate the undesirable effects or to enhance the desirable one. Whereas, Structronics are largely improves the working performance and lifetime of devices that construct from it (Badri and Al-Kayiem, 2012a; Bassiouny, 2006). Several accurate solutions of structronics shell have been presented using 3-D and 2-D theories or the discrete layer approaches. The exact closed-form solutions for multilayered piezo-electric-magnetic and purely elastic plates have been proved for special cases of Pan's analysis. Heyliger and Pan (2004) demonstrated the free vibration analysis of the simply supported and multilayered Magneto-electro-elastic (MEE) plates under cylindrical bending.

Then, Heyliger *et al*. (2004) studied two cases of the MEE plates subjected to static fields, one under cylindrical bending and the other of completely traction-free under surface potentials. Following up the previous Stroh formulation, Pan and Han (2005) presented the 3-D solutions of multilayered Functionally Graded (FG) and MEE plates. Wang *et al*. (2003) proposed a modified state vector approach to obtain 3-D solutions for MEE laminates, based on the mixed formulation of solid mechanics.

By an asymptotic approach, Tsai *et al*. (2008) studied 3-D static and dynamic behavior of doubly curved FG-MEE shells under the mechanical load, electric displacement and magnetic flux by considered the edge boundary conditions as full simple supports.

In comparison with the recently development of smart shell it could be said that the literature dealing with theoretical work in piezolaminated shell concerning coupled field phenomena in general and in magneto-thermo-electro-elastic (MTEE) in particular, is rather scarce, especially for shear deformation studies.

In this study, a fundamental theory of piezolaminates shell/plates based on the First-order Transversely Shear Deformations Theory (FSDT) will be developed. New

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issues elicited by the structural lamination, such as the distributions of center deflection over the thickness of shell are addressed.

The results supplied herein are expected to provide a foundation for the investigation of the interactive effects among the thermal, magnetic, electric and elastic fields in thin-walled structures and of the possibility to apply the MTEE adapting.

FOUNDATIONS THEORY

In order to be reasonably self-contained, in what follows, here will summarize the fundamental physical laws that govern the conservation law of electro-magnetic field and they are:

- **Faraday's law:**

$$\mathrm{curl}\,\xi = -\overset{v}{\mathcal{G}}, \rightarrow \frac{a}{at}\int_{\Omega_0} \mathcal{G}.ds$$

- **Ampere's law:**

$$\mathrm{curl}\,x = -\overset{v}{J}, \rightarrow \frac{a}{at}\int_{\Omega_0} \varepsilon.ds$$

- **Gauss's law:**

$$\mathrm{div}\,\varepsilon = F^{\varepsilon}, \overset{v}{\rightarrow}\int_{\Omega_0} F^{\varepsilon}.dv = \int_{\Omega_0} \varepsilon.ds$$

- **Conservation of flux:**

$$\mathrm{div}\,\mathcal{G} = F^{\mathcal{G}}, \overset{v}{\rightarrow}\int_{v} F^{\mathcal{G}}.dv = \int_{\Omega_0} \mathcal{G}.ds$$

THEORY OF VARIATIONAL PRINCIPLE

The energy functional are important for their use in approximate methods as well as deriving a consistent set of equations of motion coupled with free charge equations and its boundary conditions (Reddy, 1984; Bao, 1996; Tzou *et al*., 2004; Badri and Al-Kayiem, 2012b). In summary, the total energy of a shell element is defined as:

$$\delta\int_{t_0}^{t_1}[K - P]dt = 0 \tag{1}$$

where, P is total potential energy:

$$\begin{aligned} P = &\iiint_v \left[Q\,(\delta_i, \varepsilon_j, \mathcal{G}_l, \mathcal{T}) + (\mathcal{T}_{\mathcal{T}})\right]dV \\ &- \iint_{\Omega_0}(t(\delta_i, \varepsilon_j, \mathcal{G}_l) + w(\delta_i, \varepsilon_j, \mathcal{G}_l)) \end{aligned} \tag{2}$$

where, Q (δ_i, ε_i, g_1, T), t (δ_i, ε_i, g_1) and W (δ_i, ε_i, g_1) are the thermodynamic potential "Gibbs free energy", tractions and the work done by body force, electrical and magnetic charge, respectively. Moreover, the kinetic energy is:

$$K = \frac{1}{2}\iiint_v[\dot{u}^2 + \dot{v}^2 + \dot{w}^2]dV \tag{3}$$

Substituting Eq. 2 and 3 into Eq. 1 yields:

$$\begin{aligned} &\int_{t_0}^{t_1}\frac{1}{2}\iint_v[\dot{u}^2 + \dot{v}^2 + \dot{w}^2]dvdt \\ &-\Big\{\int_{t_0}^{t_1}\iiint_v[\delta Q\,(\delta_i, \varepsilon_j, \mathcal{G}_l, \mathcal{T}) + (\mathcal{T}\,\delta\tau)]dVdt \\ &-\int_{t_0}^{t_1}\iint_{\Omega_0}(\delta t\,(\delta_i, \varepsilon_j, \mathcal{G}_l) + \delta W\,(\delta_i, \varepsilon_j, \mathcal{G}_l))\,dAdt\Big\} \end{aligned}$$

The kinetic energy of the shell can be expressed as:

$$\begin{aligned} K = &\frac{1}{2}\iint_{\Omega_0}\int_{-h/2}^{h/2}\begin{pmatrix}[\dot{u}_0^2 + \dot{v}_0^2 + \dot{w}_0^2] \\ +\zeta^2[\dot{\psi}_0^2 + \dot{\psi}_\beta^2] \\ +2\zeta[\dot{u}_0^2\dot{\psi}_\alpha^2 + \dot{v}_0^2\dot{\psi}_\beta^2]\end{pmatrix} \\ &\times\left(1 + \frac{\zeta}{R_\alpha}\right)\left(1 + \frac{\zeta}{R_\beta}\right)ABd\zeta dA \end{aligned} \tag{4}$$

Based on the conservation laws of electro-magnetic field, the linear thermodynamic potential energy Q for quasi-static infinitesimal reversible system, subject to mechanical, electric, magnetic and thermal influences from its surroundings, can be approximated by:

$$Q\,(\delta_i, \varepsilon_j, g_1, \mathcal{T}) \cong \frac{1}{2}(\delta_{ij}\,\varepsilon_{kl} - \varepsilon_n\xi_n - g_q x_q - \mathcal{T})$$

Means that δ_{ii}, ε_k, $\mathcal{G}_1$ and $\mathcal{T}$ are the dependent variables of Q, while ε_{ii}, ξ_k, x_1 and $\mathcal{T}$ are the natural independent variables. In order to obtain the thermodynamic potential for which these variables are natural, is performed (Perez-Fernandez *et al*., 2009), that is:

$$\begin{aligned} 2Q = &\varsigma_{ijkl}^{\varepsilon,G,\mathcal{T}}\,\varepsilon_{ij}\varepsilon_{kl} - \varepsilon_{mn}^{\delta,G,\mathcal{T}}\,\xi_m\xi_n - \mu_{pq}^{\delta,\varepsilon,\mathcal{T}}\,\chi_p\chi_q - \theta^{\delta,\varepsilon,G}\tau^2 \\ &- 2\varrho_{mkl}^{G,\mathcal{T}}\,\varepsilon_{kl}\xi_m - 2\kappa_{pkl}^{\varepsilon_n}\,\varepsilon_{kl}\chi_p - 2\lambda_{kl}^{\varepsilon,G}\,\varepsilon_{kl}\tau - 2\eta_{pn}^{\delta,\mathcal{T}}\,\xi_n\chi_p \\ &- 2\rho_n^{\delta,G}\,\xi_n\,\tau - 2\gamma_q^{\delta,\varepsilon}\,\chi_q\,\tau \end{aligned}$$

where, Q is commonly known as Gibbs free energy, the superscripts indicate that the magnitudes must be kept constant when measuring them in the laboratory frame. The constitutive relations can be expressed formally by differentiation of Q corresponding to each dependent variable as:

$$
\begin{aligned}
\varsigma_{ij} &= \left(\frac{\partial Q}{\partial \varepsilon_{kl}}\right) = \varsigma_{ijkl}^{\varepsilon,G,T}\ \varepsilon_{ij} - \varrho_{mkl}^{G,T}\ \xi_m - \kappa_{pkl}^{\varepsilon,T}\chi_p - \lambda_{kl}^{\varepsilon,G}\tau \\
\varepsilon_k &= \left(\frac{-\partial Q}{\partial \xi_n}\right) = \varrho_{ijn}^{G,T}\ \varepsilon_{ij} + \in_{mn}^{\delta G,T}\ \xi_m + \eta_{pn}^{\delta,T}\chi_p + \rho_n^{\delta G}\tau \\
G_l &= \left(\frac{-\partial Q}{\partial \chi_q}\right) = \kappa_{ijq}^{\varepsilon,T}\ \varepsilon_{ij} + \eta_{mq}^{\delta,T}\ \xi_m + \mu_{pq}^{\delta,\varepsilon,T}\chi_p + \gamma_q^{\delta,\varepsilon}\tau \\
T &= \left(\frac{-\partial Q}{\partial_T}\right) = \lambda_{ij}^{\varepsilon,G}\ \varepsilon_{ij} + \rho_m^{\delta,G}\ \xi_m + \gamma_p^{\delta,\varepsilon}\chi_p + \theta^{\delta,\varepsilon,G}\tau
\end{aligned}
\tag{5}
$$

Then the total thermodynamic potential is given by:

$$
T = \frac{\partial Q}{\partial \varepsilon}.\ \delta\varepsilon - \frac{\partial Q}{\partial \xi}.\ \delta\xi - \frac{\partial Q}{\partial \chi}.\ \delta\chi - \frac{\partial Q}{\partial \tau}.\ \delta\tau \tag{6}
$$

While the tractions are:

$$
t(S_i, \varepsilon j, G_i) = \begin{pmatrix} (\tilde{S}_{nn}\ \delta u_n + \tilde{S}_{nt}\delta v_t + \tilde{S}_{n\zeta}\ \delta w_t) \\ +(\tilde{\varepsilon}_{nn}\delta\varphi + \tilde{\varepsilon}_{nt}\delta\varphi) \\ +(\tilde{G}_{nn}\delta\varphi + \tilde{G}_{nt}\delta\vartheta) \end{pmatrix} \tag{7}
$$

Moreover, the external study is:

$$
W(S_i, \varepsilon j, G_i) = \begin{pmatrix} F_\alpha^s u_o + F_\beta^s v_o + F_n^s w_o \\ +C_\alpha^s \psi_\alpha + C_\beta^s \psi_\beta + F^\varepsilon \varphi_o \\ +C^\varepsilon \varphi_1 + F^G \vartheta_0 + C^G \vartheta_1 \end{pmatrix} \tag{8}
$$

where, F^S_α, F^S_R and F^S_n, are the distributed forces in α, β and ζ directions, respectively and C^s_α and C^S_R are the distributed couples about the middle surface of the shell. In addition F^ε, C^ε, F^ϑ and C^ϑ are the distributed forces and couples due to electrical and magnetic charge.

Substituting Eq. 6-8 in Eq. 2 and equating the resulted equation with Eq. 1, yields the equations of motion of piezolaminated shell as shown in Eq. 9 below.

Note that, the kinetic relations (i.e., the force and moment resultants per unit length at the boundary Ω) are obtained by integrating the stresses over the shell thickness as in Eq. 10.

$$
\int_{t_0}^{t_1}\iint_{\Omega_0} \delta \begin{pmatrix} \frac{\bar{I}_1}{2}[\dot{u}_0^2 + \dot{v}_0^2 + \dot{w}_0^2] \\ \frac{\bar{I}_3}{2}[\dot{\psi}_\alpha^2 + \dot{\psi}_\beta^2] \\ \bar{I}_2[\dot{u}_0^2\dot{\psi}_\alpha^2 + \dot{v}_0^2\dot{\psi}_\beta^2] \end{pmatrix} AB\,dA\,dt - \left\{ \int_{t_0}^{t_1}\iiint_{v} \begin{pmatrix} [\varsigma_{ij}\varepsilon - \varrho_{ij}\xi - \kappa_{ij}\chi - \lambda_i\tau]\delta\varepsilon \\ -[\varrho_{ij}\varepsilon - \in_{ij}\xi - \eta_{ij}\chi + \rho_i\tau]\delta\xi \\ -[\kappa_{ij}\varepsilon - \eta_{ij}\xi + \mu_{ij}\chi + \gamma_i\tau]\delta\chi \\ -[\lambda_i\varepsilon - \rho_i\xi + \gamma_i\chi + \theta\tau]\delta\tau \end{pmatrix} dV\,dt \right.
$$

$$
\left. -\int_{t_0}^{t_1}\iint_{\Omega_0} \left(\begin{Bmatrix} \tilde{S}_{nn}(\delta u_{on} + \zeta\delta\psi_{an}) + \tilde{S}_{nt}(\delta v_{ot} + \zeta\delta\psi_{\beta t}) + \tilde{S}_{n\zeta}\delta w_t) \\ -((\tilde{\varepsilon}_{nn}(\delta\varphi_0 + \zeta\delta\varphi_1)) - (\tilde{G}_{nn}(\delta\theta_0 + \zeta\delta\theta_1)) \\ -((\tilde{\varepsilon}_{nt}(\delta\varphi_0 + \zeta\delta\varphi_1)) - (\tilde{G}_{nt}(\delta\theta_0 + \zeta\delta\theta_1)) \end{Bmatrix} + \begin{Bmatrix} [F_\alpha^s u_0 + F_\beta^s v_0 + F_n^s w_0 + C_\alpha^s\psi_\alpha + C_\beta^s\psi_\beta] \\ -[F^\varepsilon\varphi_0 + C^G\theta_1]] \end{Bmatrix} \right) AB\,dA\,dt \right\} = 0 \tag{9}
$$

Not that, the temperature τ is a known function of position. Thus, temperature field enter the formulation only through constitutive equations. While I_1, I_2 and I_3 are, the inertia terms and they define as:

$$
\bar{I}_j = \left[I_j + I_j + 1\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right) + \frac{I_{j+2}}{R_\alpha R_\beta} \right] \text{for } j = 1, 2, 3
$$

And:

$$
[I_1, I_2, I_3, I_4, I_5] = \sum_{k=1}^{N}\int_{h_k-1}^{h_k} I^k (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5) d\zeta
$$

where, (I^k) is the mass density of the kth layer of the shell per unit midsurface area. While the energy expressions described above are used to derive the equations of motion:

$$
\begin{Bmatrix} N_\alpha^S & M_\alpha^S \\ N_\beta^S & M_\beta^S \\ Q_\alpha^S & P_\alpha^S \\ Q_\beta^S & P_\beta^S \\ N_{\alpha\beta}^S & M_{\alpha\beta}^S \\ N_\alpha^\varepsilon & M_\alpha^\varepsilon \\ N_\beta^\varepsilon & M_\beta^\varepsilon \\ N_\alpha^G & M_\beta^G \\ N_\beta^G & M_\beta^G \end{Bmatrix} = \int_{-n/2}^{n/2} (1, \varsigma) \begin{Bmatrix} \delta_\alpha \\ \delta_\beta \\ \delta_{\beta\varsigma} \\ \delta_{\alpha\varsigma} \\ \delta_{\alpha\beta} \\ \varepsilon_\alpha \\ \varepsilon_\beta \\ G_\alpha \\ G_\beta \end{Bmatrix} d\varsigma \tag{10}
$$

Also, can rewrite Eq. 10 in term of constitutive relations Eq. 5 directly as that expressed below in Eq. 11.

Thus, the constitutive terms in Eq. 9 could be replaced by the kinetic relations Eq. 11 for a reason of casting the equation of motion to be dependent of forces and moment resultant as well as to reduce the volume integral to double integral.

By recasting Eq. 9 to put in the familiar form, the governing equations of motion and the equation charge equilibrium for first-order shearable deformation case could be derived based on the fundamental Lemma of calculus of variations; e.g., by integrating the field gradients by parts to relieve the virtual fields and setting its coefficients to zero individually.

EQUATIONS OF MOTION

In order to solve the resulted equation of motion, we introduce the following assumptions to cast the equation of motion in thick (or shear deformation) shell theories. Furthermore, the deepness (or shallowness) of the shell, is also one criterion used in developing shell equations (Badri and Al-Kayiem, 2012c).

Thus, shell is referred to as a shallow, when it has infinity $R_{\alpha\beta}$ and the term $(1+\zeta/R_i) = 1$: Where R_i is either of the curvature parameter R_α, R_β, or $R_{\alpha\beta}$ (Qatu, 2004). If it is represented by the plane coordinate systems for the case of rectangular orthotropy, this leads to constant Lame parameters (i.e., A, B = 1). In additional, the radii of curvature are assumed very large compared to the in-plane displacements. i.e., $u_i/R_i = 0$, where $i = \alpha,\beta$ and α, $u_i = u_o$, or v_o.

Hence, the procedure outlined above, is valid irrespective of using the Navier solution. The Navier-type solution can be applied to obtain exact solution as $(K_{ii} + \lambda^2 M_{ii})$ $\{\Delta\} = \{F\}$, which is an eigenvalue problem. For nontrivial solution, the determinant of the matrix in the parenthesis is set to zero. Then the configuration of K_{ii} terms for SS-1, cross-ply and rectangular plane form is listed in the Appendix.

RESULTS AND DISCUSSION

To prove the validity of the developed theory, laminated composite square plate (a/b = 1) with both the upper and lower surfaces embedded smart materials is considered. The plate structure considered here is made of $BaTiO_3$ and $CoFe_2O_4$ composite material. The material properties are given in several papers like (Badri and Al-Kayiem, 2011a-c) and it will not be included here.

First, the example of sandwich piezoelectric and magnetostrictive plate that studied and analyzed exactly by various researchers e.g., Pan and Heyliger (2001) and Chen *et al.* (2005) is considered here for validation and comparison. Table 1 gives the lowest five frequency parameters:

$$\Omega = wa^2 \sqrt{\rho_{max} / \delta_{max}{}_{h^2}}$$

of the fundamental vibrational mode (m = n = 1) which is of practical importance (Pan and Heyliger, 2002), whereas, δ_{max} being the maximum of the δ_{ii} in the whole sandwich plate and $\rho_{max} = 1$, which was defined by Pan and Heyliger (2002) and adopted by Chen *et al.* (2005). While in Table 1 it clearly seen that the frequency results obtained by the present model are in close agreement with those obtained by Chen *et al.* (2005) using alternative state space formulations:

$$\begin{bmatrix} \begin{Bmatrix} N^{\delta}_{ky} & M^{\delta}_{ky} \\ Q^{\delta}_{ky} & P^{\delta}_{ky} \end{Bmatrix} \\ \{N^{\varepsilon}_{ny} \quad M^{\varepsilon}_{ny}\} \\ \{N^{\varsigma}_{qy} \quad M^{\varsigma}_{qy}\} \\ \end{bmatrix} = \int_{-h/2}^{h/2} (1,\varsigma) \begin{pmatrix} [\varsigma^{\varepsilon,\varsigma\,T}_{ijkl}\,\varepsilon_{ij} - Q^{\varsigma\,T}_{mkl}\,\xi_m - \kappa^{\varepsilon,T}_{pkl}\,X_p - \lambda^{\varepsilon,\varsigma}_{kl}\,\tau] \\ [Q^{\varsigma\,T}_{ijn}\,\varepsilon_{ij} + \epsilon^{\delta,\varsigma,T}_{mkl}\,\xi_m + \eta^{\varepsilon,T}_{pn}\,X_p + \gamma^{\varepsilon,\varsigma}_{q}\,\tau] \\ [\kappa^{\varepsilon\,T}_{ijq}\,\varepsilon_{ij} + \eta^{\delta,T}_{mq}\,\xi_m + \mu^{\varepsilon,\varepsilon,T}_{pq}\,X_p + \gamma^{\varepsilon,\varepsilon}_{q}\,\tau] \end{pmatrix} \left(1 + \frac{\zeta}{R_x}\right) d\zeta \begin{cases} \text{For}(x,y) = (\alpha,\beta) \\ \text{and}\, x \neq y \end{cases} \quad (11)$$

Note that Table 1 shows the frequencies of the first class of vibration only. It is worth to highlight that the 1st mode of vibration shows 100% agreement with literature, while the discrepancy in other higher modes are negligible in practical sense. Further results and conclusions about the classes of vibration can be found by Pan and Heyliger (2001) and Chen *et al.* (2005) for $BaTiO_2/CoFe_2O_4$ sandwich plate. In fact, those results have been successfully reproduced and discrepancy around 5% is observed.

It should be mentioned here that present model has been verified for results available in literature for pure elastic shell by letting Q_{ii} and/or κ_{ii} equal to zero and rigorous agreement was found.

While the bonded error for plate results were predicted and explained as due to the assumption of specialization of shell theory to plate by letting $R_\alpha = R_\beta = R_{\alpha\beta} = \infty$. In essence the plate can be regarded as a special case of the present analysis, but in fact it has a purpose of verification with literature only. In the other hand, Fig. 1 shows the center deflections ϖ, angle of twist ψ_α and ψ_β, in-plane displacement u and v, electrical potential φ and magnetic potential θ sensory responses for sandwich shell formed from three smart layers. It is perceived that the elastic deflections, electrical potential and magnetic potential have similar occurrence.

Table 1: Comparison of recent results of the lowest 5 frequency parameters of the sandwich plate with results of Pan and Heyliger (2002)

	P only		M only		P/M/P*		M/P/M	
Order	Ref.	Present†	Ref.	Present	Ref.	Present	Ref.	Present
1	2.30033	2.3003	1.97472	1.9747	1.82648	1.8366	1.89865	1.7474
2	2.80145	3.3011	2.33726	2.7774	2.15561	2.8999	2.31557	2.7622
3	3.93927	4.2475	3.18631	3.5198	3.07652	4.0917	3.11555	3.9047
4	5.31985	4.9574	4.23897	4.0664	4.11470	5.3178	4.17674	5.0882
5	6.79683	5.5112	5.37695	4.5048	5.24651	6.5492	5.30704	6.2878

† Note that the present results are for shell of (R_α, R_β, $R_{\alpha,\beta} = \infty$) and the shear correction factor used in FSDT is ($\kappa^2 = 5/6$), *(P/M/P) is denoting Piezoelectric [inner]/Magnetostrictive [middle]/Piezoelectric [outer]

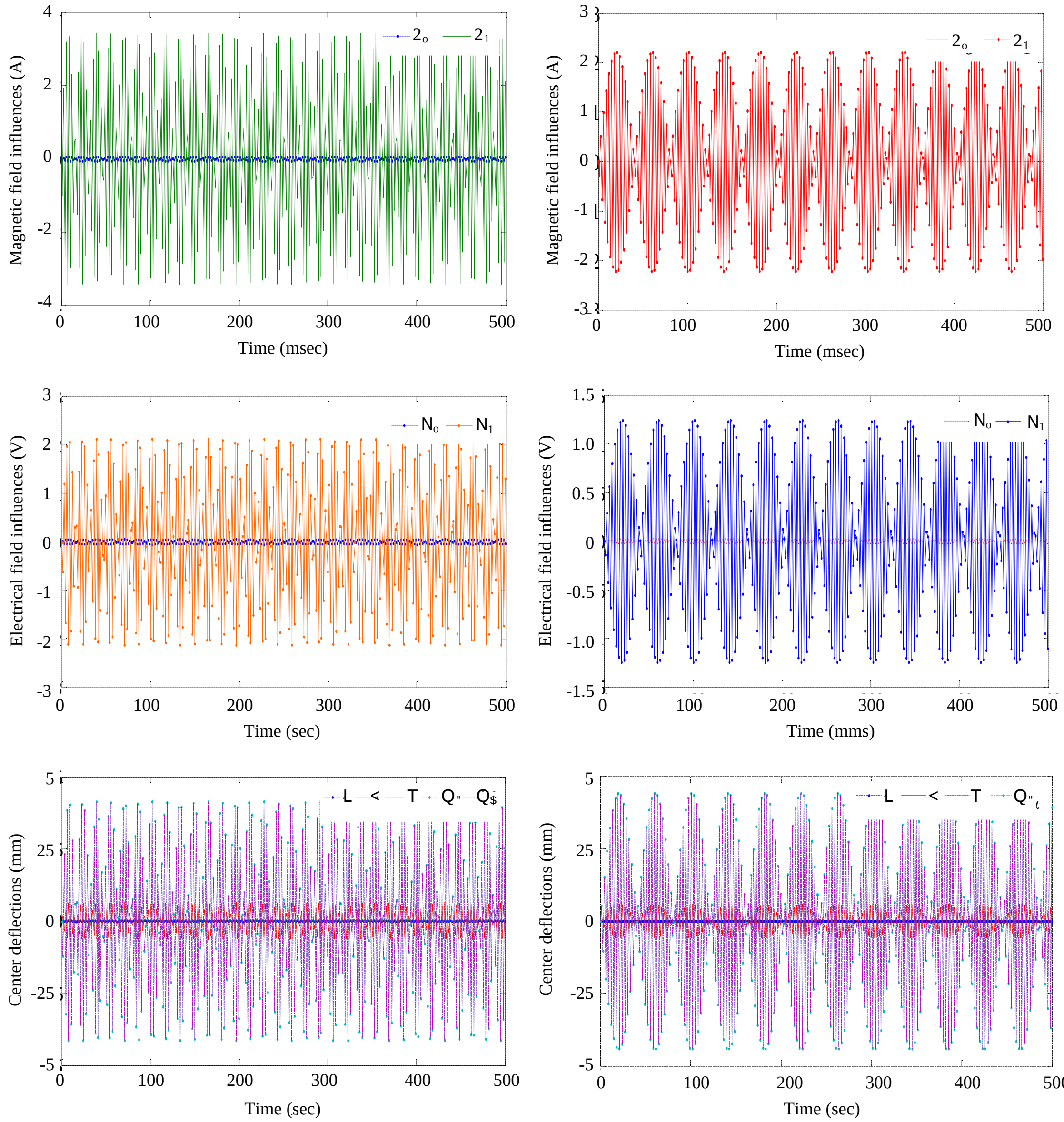

Fig. 1: The uncontrolled responses of laminated composite spherical shell of at which the right side is representing the P/M/P scheme when the left side is being for M/P/M

It is interesting to note, that the sensory responses have simple discriminate behavior against the variation in the shell dimensions.

CONCLUSION

In this study a model is developed for dynamic analysis of piezolaminated shell structure and embedded smart material lamina and influenced by MTEE load. The fundamental theory is derived based on FSDT involving Codazzi-Gauss geometrical discretion. The theory is casted in version of piezolaminated plate of rectangular plane-form (for purpose of validation and verification only). At which the generic forced-solution procedures for the response were derived and its frequency parameters were evaluated in simply supported boundary condition.

Results have shown a close agreement with those obtained by Chen *et al*. (2005). Furthermore, the present model has been verified for results available in literature for pure elastic shell by assuming Q_{ii} and/or κ_{ii} equal to zero and rigorous agreement was also found.

The present results may serve as a reference in developing the piezolaminated shell theories and to improve the benchmark solutions for judging the existence of imprecise theories and other numerical approaches.

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APPENDIX

$$K_{11}=-\bar{\varsigma}_{11}^{1}\,\alpha_m^2-\tilde{\varsigma}_{66}^{1}\,\beta_n^2,\ K_{12}=-(\varsigma_{12}^{1}+\varsigma_{66}^{1})\alpha_m\beta_n$$

$$K_{22}=-\bar{\varsigma}_{22}^{1}\,\beta_n^2-\bar{\varsigma}_{66}^{1}\,\alpha_m^2,\ K_{13}=\left(\frac{\bar{\varsigma}_{11}^{1}}{R_\alpha}+\frac{\bar{\varsigma}_{12}^{1}}{R_\beta}\right)\alpha_m$$

$$K_{23}=\left(\frac{\varsigma_{12}^{1}}{R_\alpha}+\frac{\tilde{\varsigma}_{22}^{1}}{R_\beta}\right)\beta_n,\ K_{35}=\left(-\tilde{\varsigma}_{44}^{1}+\frac{\varsigma_{12}^{2}}{R_\alpha}+\frac{\tilde{\varsigma}_{12}^{2}}{R_\beta}\right)\beta_n$$

$$K_{33}=-\tilde{\varsigma}_{44}^{1}\beta_n^2-\tilde{\varsigma}_{55}^{1}\alpha_m^2-\left(\frac{\bar{\varsigma}_{11}^{1}}{R_\alpha^2}+\frac{2\varsigma_{12}^{1}}{R_\alpha R_\beta}+\frac{\tilde{\varsigma}_{22}^{1}}{R_\beta^2}\right)$$

$$K_{14}=-\bar{\varsigma}_{11}^{2}\,\alpha_m^2-\tilde{\varsigma}_{66}^{2}\beta_n^2,\ K_{24}=-(\varsigma_{12}^{2}+\varsigma_{66}^{2})\alpha_m\beta_n$$

$$K_{34}=\left(-\bar{\varsigma}_{55}^{1}+\frac{\bar{\varsigma}_{11}^{2}}{R_\alpha}+\frac{\varsigma_{12}^{2}}{R_\beta}\right)\alpha_m$$

$$K_{34}=-\bar{\varsigma}_{55}^{1}-\bar{\varsigma}_{11}^{3}\alpha_m^2-\tilde{\varsigma}_{66}^{3}\beta_n^2$$

$$K_{15}=-(\varsigma_{12}^{2}+\varsigma_{66}^{2})\alpha_m\beta_n,\ K_{25}=\tilde{\varsigma}_{22}^{2}\beta_n^2-\bar{\varsigma}_n^{2}-\bar{\varsigma}_{66}^{2}\alpha_m^2$$

$$K_{55}=-\tilde{\varsigma}_{44}^{1}-\bar{\varsigma}_{66}^{3}\alpha_m^2-\tilde{\varsigma}_{22}^{3}\beta_n^2$$

$$K_{45}=-(\varsigma_{12}^{3}+\varsigma_{66}^{3})\alpha_m\beta_n$$

$$K_{16}=-\left(\frac{\varrho_{14}^{1}}{R_{\alpha\beta}}+\frac{\bar{\varrho}_{15}^{1}}{R_\alpha}\right)\alpha_m,\ K_{26}=\left(\frac{\varrho_{24}^{1}}{R_\beta}+\frac{\varrho_{25}^{1}}{R_{\alpha\beta}}\right)\beta_n$$

$$K_{36}=-(\varrho_{15}^{1}\alpha_m^2+\varrho_{24}^{1}\beta_n^2),\ K66=-(\bar{\varepsilon}_{11}^{1}\alpha_m^2+\tilde{\varepsilon}_{22}^{1}\beta_n^2)$$

$$K_{46}=\left(-\varrho_{15}^{1}+\frac{\bar{\varrho}_{15}^{2}}{R_\alpha}\right)\alpha_m,\ K_{56}=\left(-\tilde{\varrho}_{24}^{1}+\frac{\tilde{\varrho}_{24}^{2}}{R_\beta}\right)\beta_n$$

$$K_{17}=\left(\frac{\varrho_{14}^{2}}{R_{\alpha\beta}}+\frac{\bar{\varrho}_{15}^{2}}{R_\alpha}\right)\alpha_m,\ K_{27}=\left(\frac{\tilde{\varrho}_{24}^{2}}{R_\beta}+\frac{\varrho_{25}^{2}}{R_{\alpha\beta}}\right)\beta_n$$

$$K_{37}=-(\bar{\varrho}_{15}^{2}\alpha_m^2+\tilde{\varrho}_{24}^{2}\beta_n^2),\ K_{47}=\left(\bar{\varrho}_{15}^{2}+\frac{\bar{\varrho}_{15}^{2}}{R_\alpha}\right)\alpha_m$$

$$K_{67}=-(\bar{\varepsilon}_{11}^{2}\alpha_m^2+\bar{\varepsilon}_{22}^{2}\beta_n^2),\ K_{77}=-(\bar{\varepsilon}_{11}^{3}\alpha_m^2+\bar{\varepsilon}_{22}^{3}\beta_n^2)$$

$$K_{67}=\left(\frac{\kappa_{14}^{1}}{R_{\alpha\beta}}+\frac{\kappa_{15}^{1}}{R_\alpha}\right)\alpha_m,\ K_{28}=\left(\frac{\tilde{\kappa}_{14}^{1}}{R_\beta}+\frac{\kappa_{15}^{1}}{R_{\alpha\beta}}\right)\beta_n$$

$$K_{38}=(\bar{\kappa}_{15}^{1}\alpha_m^2+\tilde{\kappa}_{24}^{1}\beta_n^2),\ K_{48}=-\left(\bar{\kappa}_{15}^{1}+\frac{\bar{\kappa}_{15}^{2}}{R_\alpha}\right)\alpha_m$$

$$K_{57}=\left(\varrho_{24}^{2}+\frac{\tilde{\varrho}_{24}^{2}}{R_\beta}\right)\beta_n,\ K_{58}=\left(\tilde{\kappa}_{24}^{1}+\frac{\tilde{\kappa}_{24}^{1}}{R_\beta}\right)\beta_n$$

$$K_{68}=-\left(\bar{\eta}_{11}^{1}\alpha_m^2+\tilde{\eta}_{22}^{1}\beta_n^2\right)$$

$$K_{78}=-\left(\bar{\eta}_{11}^{1}\alpha_m^2+\tilde{\eta}_{22}^{1}\beta_n^2\right)$$

$$K_{88}=-\left(\bar{\mu}_{11}^{1}\alpha_m^2+\bar{\mu}_{22}^{1}\beta_n^2\right),\ K_{19}=\left(\frac{\kappa_{14}^{2}}{R_{\alpha\beta}}+\frac{\kappa_{15}^{2}}{R_\alpha}\right)\alpha_m$$

$$K_{29}=\left(\frac{\tilde{\kappa}_{24}^{2}}{R_\beta}+\frac{\kappa_{25}^{2}}{R_{\alpha\beta}}\right)\beta_n,\ K_{29}=-(\bar{\kappa}_{15}^{2}\alpha_m^2+\tilde{\kappa}_{24}^{2}\beta_n^2)$$

$$K_{49}=\left(-\bar{\kappa}_{15}^{2}\frac{\kappa_{25}^{3}}{R_\alpha}\right)\alpha_m,\ K_{59}=\left(-\bar{\kappa}_{24}^{2}+\frac{\tilde{\kappa}_{24}^{3}}{R_\beta}\right)\beta_n$$

$$K_{69}=(\bar{\eta}_{11}^{2}\alpha_m^2+\bar{\eta}_{22}^{2}\beta_n^2),\ K_{79}=(\bar{\eta}_{11}^{3}\alpha_m^2+\bar{\eta}_{22}^{3}\beta_n^2)$$

$$K_{89}=(\bar{\mu}_{11}^{2}\alpha_m^2+\tilde{\mu}_{22}^{2}\beta_n^2),\ K_{99}=(\bar{\mu}_{11}^{3}\alpha_m^2+\tilde{\mu}_{22}^{3}\beta_n^2)$$

NOMENCLATURE

Latin symbols:

a, b	=	Length and width of the shell in (m)
A_i, A,B	=	Lame' parameter
ε	=	Electric displacement
F^{ε}	=	Electrical charge density
F^{ς}	=	Electric current density or magnetic charge density
F^{T}	=	Thermal forces resultant
F_i^{δ}	=	Elastic body forces in (N)
ς	=	Magnetic inductions
K	=	Kinetic energy
ñ	=	Unit outward normal
$N_i, Q_i, \tilde{N}^{\delta,\varepsilon,\varsigma,T}$	=	Edge forces, shear forces and its free traction resultant
$M_i, P_i, \tilde{M}_i^{\delta,\varepsilon,\varsigma,T}$	=	Edge moment, higher shear terms and its free traction resultant
$Q^{\delta,\varepsilon,\varsigma,T}$	=	Gibbs free energy in (Joule)
u_o, v_o, w_o	=	Mid-surface displacements of the shell
R_i,	=	Radius of curvature
$\delta_{ij}, \tilde{\delta}_i$	=	Stress field and Free elastic tractions, respectively in (Pa)
t	=	Surface tractions
T	=	Thermal Gain (Entropy)
W	=	Work (body forces) in (N m)

Greek symbols

α, β, ζ	=	Curvilinear coordinates, α and β for the reference surface and ζ for the normal axis
$\gamma_q^{\delta,\varepsilon}$	=	Thermo-magnetic
$\varepsilon_{i,j}$	=	Elastic strain
$\eta_{pn}^{\delta,T}$	=	Magneto-electric
$\theta^{\delta,\varepsilon,\varsigma}$	=	Thermal properties (Pa/°C^2)
$\kappa_{pkl}^{\varepsilon,T}$	=	Magneto-elastic
$\lambda_{kl}^{\varepsilon,\varsigma}$	=	Thermo-elastic
$\mu_{pq}^{\varepsilon,\varepsilon,T}$	=	Magnetic (Ns^2/C^2)
ζ	=	Electric field
$\rho_n^{\delta,\varsigma}$	=	Thermo-electric
ρ_o	=	Density
$\zeta_{ijkl}^{\delta,\varsigma,T}$	=	Elastic properties in (pa)
τ	=	Thermal Field
$\in_{mn}^{\delta,\varsigma,T}$	=	Electric properties in (C^2/Nm^2)
$Q_{mkl}^{\varsigma,T}$	=	Electro-elastic
φ	=	Electric potential

x	=	Magnetic field vector
ϑ	=	Magnetic potential
ψ_α, ψ_β	=	Mid-surface rotations

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